ON A QUESTION OF BRENDLE AND FARB

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ABSTRACT. In this note we prove that there is no constant C, depending on the genus of the surface, such that every element in the mapping class group can be written as a product of at most C torsion elements, answering a question of T. E. Brendle and B. Farb in the negative.

1. Introduction

In a recent paper [1], Tara E. Brendle and Benson Farb showed that the mapping class group of a closed orientable surface of genus $g \geq 3$ is generated by three torsion elements, and also by seven involutions. They asked whether there is a number $C = C(g)$ such that every element in the mapping class group can be written as a product of at most C torsion elements. The purpose of this note is to give a negative answer to this question. We deduce our result from the classification of torsion elements in the mapping class group and the fact that the mapping class group is not uniformly perfect, although it is perfect.

2. Preliminaries

Let S be a closed orientable surface of genus g and let Mod_g denote the mapping class group of S, the group of the isotopy classes of orientation-preserving diffeomorphisms $S \to S$.

The following theorem is due to Kazuo Yokoyama [6]. In fact, Yokoyama gives the number of conjugacy classes explicitly.

Theorem 1. *The number of conjugacy classes of torsion elements in the mapping class group* Mod_g *is finite.*

Corollary 2. *Suppose that* $g \geq 3$. *There is a constant* $T(g)$ *such that every torsion element in the mapping class group* Mod_g *can be written as a product of at most* $T(g)$ *commutators.*

Proof. Choose a representative from each conjugacy classes and write them as a product of commutators. Let $T(g)$ be the maximum number of such commutators. Since the conjugation of a commutator is again a commutator, the corollary follows from Theorem 1. □

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In the above corollary and its proof, we may replace “commutator(s)” by “Dehn twist(s)”.

It is well known that the mapping class group Mod_g is perfect for $g \geq 3$, but it is not uniformly perfect.

Theorem 3. ([4],[2]) *Suppose that $g \geq 3$. There is no constant $C(g)$ such that every element in the mapping class group Mod_g can be written as a product of at most $C(g)$ commutators.*

3. The results

Theorem 4. *Suppose that $g \geq 3$. There is no constant $C(g)$ such that every element in the mapping class group Mod_g of a closed orientable surface of genus g can be written as a product of at most $C(g)$ torsion elements.*

Proof. Suppose that there is such a constant $C(g)$. We then conclude from Corollary 2 that every element in Mod_g is a product of at most $C(g)T(g)$ commutators, which contradicts to Theorem 3. □

Corresponding question can also be asked for Dehn twists; is there a constant $C(g)$ such that every element in Mod_g is a product of at most $C(g)$ Dehn twists? This question, which was asked to the author by András Stipsicz, was the motivation for the paper [4]. Since every Dehn twist can be written as a product of two commutators [5], the following theorem, which is implicit in [4] and [2], follows easily from Theorem 3. We note that it also holds for the mapping class group Mod_2.

Theorem 5. *Suppose that $g \geq 3$. There is no constant $C(g)$ such that every element in the mapping class group of a closed orientable surface can be written as a product of at most $C(g)$ Dehn twists.*

4. Final comments

Any element $f \in \mathrm{Mod}_g$ can be written as a product of torsion elements [1]. Let us define $\tau_g(f)$ to be the minimum number of such torsion elements and call it the *torsion length* of f. Clearly, the sequence $\tau_g(f^n)$ is subadditive, that is, $\tau_g(f^{n+m}) \leq \tau_g(f^n)+\tau_g(f^m)$ for all positive integers n, m. Therefore, the limit

$$\lim_{n\to\infty} \frac{\tau_g(f^n)}{n}$$

exists. Let us denote this limit by $||f||_{\tau_g}$ and call it the *stable torsion length* of f.

If t_a denotes the Dehn twist about some nonseparating simple closed curve a, then t_a is a product of two torsion elements (see [1], Lemma 3). Hence, $\tau_g(t_a) = 2$. What is $\tau_g(t_a)$ if a is separating? In general, what is $\tau_g(t_a^n)$? The fact $\tau_g(t_a) = 2$ gives the upper bound $||t_a||_{\tau_g} \leq 2$.

On the other hand, for any nontrivial simple closed curve a the growth rate of t_a is linear [4, 3] and its stable commutator length is positive [4, 2]

(see the references for definitions). Inspired by these facts, the following conjecture seems reasonable.

Conjecture 1. The stable torsion length $||t_a||_{\tau_g}$ is positive.

Clearly, if f is a torsion element, then the torsion length of any power of f is either one or zero. Therefore, its stable torsion length $||f||_{\tau_g}$ is zero. We end by stating a stronger version of Conjecture 1.

Conjecture 2. If $||f||_{\tau_g} = 0$, then f is torsion.

References

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